SINGULAR EXTREMAL CONTROL PROBLEM WITH TIME DELAY

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ABSTRACT

In this thesis, the singular extremal control problem with time delay is studied from the viewpoint of the calculus of variation and matrix theory. In Chapter I, some sufficient conditions and a necessary condition for the non-negativity of $\delta^2 J$ are obtained for an optimal control problem with a delayed state vector. In Chapter II, the controllability of linear system and the normality for the problem of Bolza is studied. And in Chapter III, a local dual property of the second variation of J(u) is discussed.

It is suggested that further studies can be done in the area where there is a delay in the control variable.

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CHAPTER I

SUFFICIENT CONDITIONS AND NECESSARY CONDITION

1. Introduction

The theory of singular extremal control problem has received much attention during the last decade and extensive results are available for a wide class of problems. However, for systems containing time delays, the theory is incomplete.

In this chapter, some sufficient conditions and a necessary condition are derived for time-delay singular extremal control problems in which there is a constant delay in the state vector only. We will see that almost all the results obtained here can be extended to the case with variable delay.

2. Formulation of the Problem (c.f. [1])

The general problem under consideration is now defined, and the notations used throughout this chapter are explained.

The optimal control problem with delayed state vector is the following problem. It is desired to choose a control u which minimizes the performance criterion.

$$J(u) = \varphi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), x(t-\tau), u(t), t)\, dt \qquad (1.1)$$

subject to

$$\dot{x} = f(x(t), x(t-\tau), u(t), t) \qquad (1.2)$$

$$x(t) = x_0(t), \qquad t_0-\tau \le t \le t_0 \qquad (1.3)$$

$$\psi(x_f) = \psi(x(t_f)) = 0 \qquad (1.4)$$

where $\tau > 0$, $x_0(t)$ and $\psi(x_f)$ are given vector functions, and t_0, t_f and τ are given scalar constants.

The state x is an n-dimensional vector function of t, and $x \in D'$, i.e., x is continuous with piecewise continuous first derivatives. Both φ and L are scalars; f is an n-vector; ψ is a r-vector, and along the optimal solution the rank of ψ_{x_f} is assumed to be r . Let u be an m-vector function of t which is piecewise contiunous (PWC) with finite jump discontinuities, and $m \le n$. Assume that L and f are in class C^2 with respect to their arguments, and that φ and ψ are in class C^2 with respect to x_f.

A system described by (1.2) is to be controlled over a fixed time interval by the control u in such a manner that the initial conditions (1.3) and the terminal condition (1.4) are satisfied. This control u will be called 'admissible', and the corresponding trajectory x will also be called 'admissible'. The control which satisfies this and which minimizes the performance criterion (1.1) is called the optimal control $u^*(t)$, and the resulting path of the system state, $x^*(t)$, is called the optimal trajectory. The basic difference between this problem and the non-delay problem is that the rate of change of the system state depends not only on the present value of state and control, but also on the value of the state τ time units previous to present time. In general, the delayed state may also appear in the performance index.

Along the optimal solution (x^*,u^*) the first variation of J, denoted by $\delta J(u;\delta u)$ or $J_1(u,\eta)$, must vanish. The term $\eta = \delta u$, is the first variation of u along the optimal path. Define $H = L+p^T f$, the Hamiltonian of the sytem.

$\phi = \varphi + v^T\psi$, where p is an n-vector function of t, and v is a constant p-vector. Then,

$$J(u) = \phi(x_f) + \int_{t_0}^{t_f} \{H(x, x_\tau, u, p, t) - p^T \dot{x}\}\, dt. \qquad (x_\tau = x(t-\tau))$$

Let δu and $\delta^2 u$ be PWC vector functions defined on $[t_0, t_f]$. Let δx, $\delta^2 x$ be D^1-vector functions defined on $[t_0-\tau,\ t_f]$ and satisfying the following conditions:

$$\delta x(t) = 0, \qquad \delta^2 x(t) = 0, \qquad\qquad t_0-\tau \le t \le t_0.$$

Let $J_H(u) = \int_{t_0}^{t_f} \{H(x, x_\tau, u, p, t) - p^T \dot{x}\}\, dt$, and consider

$$J(u)=\phi(x_f)+J_H$$

$$\delta J_H=\frac{d}{d\varepsilon}J_H(u+\varepsilon\delta u+\tfrac{1}{2}\varepsilon^2\delta^2 u)\Big|_{\varepsilon=0}$$

$$=\int_{t_0}^{t_f}[H_x(\delta x+\varepsilon\delta^2 x)+H_{x_\tau}(\delta x_\tau+\varepsilon\delta^2 x_\tau)+H_u(\delta u+\varepsilon\delta^2 u)-p^T(\delta\dot{x}+\varepsilon\delta^2\dot{x})]\Big|_{\varepsilon=0}dt$$

$$=\int_{t_0}^{t_f}[H_x\delta x+H_{x_\tau}\delta x_\tau+H_u\delta u-p^T\delta\dot{x}]\,dt.$$

Denoting $\xi=\delta x$, $\xi_\tau=\delta x_\tau$, $\eta=\delta u$, then

$$\delta J(\eta)=\phi_{x_f}\delta x_f+\int_{t_0}^{t_f}[H_x\xi+H_{x_\tau}\xi+H_u\eta+\dot{p}^T\xi]\,dt-p\xi\Big|_{t_0}^{t_f}$$

$$=(\phi_{x_f}-p^T(t_f))\delta x_f+\int_{t_f-\tau}^{t_f}+\int_{t_0}^{t_f-\tau}[(H_x+\dot{p}^T)\xi+H_{x_\tau}\xi_\tau+H_u\eta]\,dt.$$

Now, we consider

$$\int_{t_0}^{t_f}H_{x_\tau}(t)\xi(t-\tau)\,dt \qquad\qquad (a)$$

let $t-\tau=s$, then $t=s+\tau$, $dt=ds$

$$(a)=\int_{t_0-\tau}^{t_f-\tau}H_{x_\tau}(s+\tau)\xi(s)\,ds=\int_{t_0}^{t_f-\tau}H_{x_\tau}(t+\tau)\xi(t)\,dt, \qquad\qquad (b)$$

substituting (b) into $\delta J(\eta)$, we have

$$\delta J(\eta)=(\phi_{x_f}-p^T(t_f))\delta x_f+\int_{t_0}^{t_f-\tau}(H_x+H_{x_\tau}|^{t+\tau}+\dot{p}^T)\xi\,dt+\int_{t_f-\tau}^{t_f}(H_x+\dot{p}^T)\xi dt+\int_{t_0}^{t_f}H_u\eta\,dt.$$

Let $\delta J(\eta)=0$, we obtain the following necessary condition for optimal solution. At points t not corresponding to corners, the Euler-Lagrange equations must be satisfied:

$$H_u=0, \qquad t_0\le t\le t_f \qquad (1.5)$$

$$\dot{p}^T=-H_x-H_{x_\tau}(t+\tau) \qquad t_0\le t\le t_f-\tau \qquad (1.6)$$

$$\dot{p}^T=-H_x, \qquad t_f-\tau\le t\le t_f. \qquad (1.7)$$

The transversality condition is:

$$p^T(t_f) = \phi_{x_f} \tag{1.8}$$

In addition, the initial and terminal conditions (1.3) and (1.4), and the state equation (1.2) must be satisfied.

If $\det H_{uu} \neq 0$ along u^*, the solution is said to be nonsingular, otherwise singular. For the remainder of this chapter, it is assumed that the optimal solution (u^*, x^*) has no corners.

3. Second Variation

The second variation along u^* is

$$\delta^2 J(u^*, \delta u) = \frac{1}{2}\frac{d^2}{d\varepsilon^2}\left[\phi(x_f+\varepsilon\delta x_f+\tfrac{1}{2}\varepsilon^2\delta^2 x_f) + J_H(u+\varepsilon\delta u+\tfrac{1}{2}\varepsilon^2\delta^2 u)\right]\Big|_{\varepsilon=0}$$

$$= \frac{1}{2}\delta x_f^T\phi_{x_f x_f}\delta x_f + \frac{1}{2}\phi_{x_f}\delta^2 x_f + \frac{1}{2}\frac{d^2}{d\varepsilon^2}J_H(u+\varepsilon\delta u+\tfrac{1}{2}\varepsilon^2\delta^2 u)\Big|_{\varepsilon=0}$$

since $\frac{1}{2}\frac{d^2}{d\varepsilon^2}J_H(u+\varepsilon\delta u+\tfrac{1}{2}\varepsilon^2\delta^2 u)\Big|_{\varepsilon=0}$

$$= \frac{1}{2}\int_{t_0}^{t_f}\{[(\delta x+\varepsilon\delta^2 x)^T H_{xx} + (\delta x_\tau+\varepsilon\delta^2 x_\tau)^T H_{x_\tau x} + (\delta u+\varepsilon\delta^2 u)^T H_{ux}](\delta x+\varepsilon\delta^2 x)$$

$$+[(\delta x+\varepsilon\delta^2 x)^T H_{xx_\tau} + (\delta x_\tau+\varepsilon\delta^2 x_\tau)^T H_{x_\tau x_\tau} + (\delta u+\varepsilon\delta^2 u)^T H_{ux_\tau}](\delta x_\tau+\varepsilon\delta^2 x_\tau)$$

$$+[(\delta x+\varepsilon\delta^2 x)^T H_{xu} + (\delta x_\tau+\varepsilon\delta^2 x_\tau)^T H_{x_\tau u} + (\delta u+\varepsilon\delta^2 u)^T H_{uu}](\delta u+\varepsilon\delta^2 u)$$

$$+H_x\delta^2 x + H_{x_\tau}\delta^2 x_\tau + H_u\delta^2 u - p^T\delta^2\dot{x}\}\Big|_{\varepsilon=0} dt$$

$$= \frac{1}{2}\int_{t_0}^{t_f}\{2W + H_x\delta^2 x + H_{x_\tau}\delta^2 x_\tau + H_u\delta^2 u - p^T\delta^2\dot{x}\}\, dt$$

where $2W(t, \delta x, \delta x_\tau, \delta u)$

$$= \delta x^T H_{xx}\delta x + 2\delta x^T H_{xx_\tau}\delta x_\tau + \delta x_\tau^T H_{x_\tau x_\tau}\delta x_\tau + 2\delta x^T H_{xu}\delta u + 2\delta x_\tau^T H_{x_\tau u}\delta u + \delta u^T H_{uu}\delta u$$

then $\delta^2 J(u^*, \delta u) = \frac{1}{2}\delta x_f^T\phi_{x_f x_f}\delta x_f + \frac{1}{2}\phi_{x_f}\delta^2 x_f + \frac{1}{2}\int_{t_0}^{t_f}\{2W + H_x\delta^2 x + H_{x_\tau}\delta^2 x_\tau + H_u\delta^2 u - p^T\delta^2\dot{x}\}\, dt$

Integrating the $\delta^2\dot{x}$ term by parts, changing variables in the $\delta^2 x_\tau$ term, and noting that $\delta^2 x=0$ on $[t_0-\tau,t_0]$,

we have

$$\delta^2 J(u^*,\delta u)=\frac{1}{2}\delta x_f^T\phi_{x_f x_f}\delta x_f+\frac{1}{2}\phi_{x_f}\delta^2 x_f+\frac{1}{2}\int_{t_0}^{t_f}2W dt-\frac{1}{2}[p^T\delta x]\Big|_{t_0}^{t_f}$$

$$+\frac{1}{2}\int_{t_0}^{t_f-\tau}[(H_x+H_{x_\tau}(t+\tau)+\dot{p}^T)\,\delta^2 x]\,dt+\frac{1}{2}\int_{t_f-\tau}^{t_f}[(H_x+\dot{p}^T)\,\delta^2 x]\,dt+\frac{1}{2}\int_{t_0}^{t_f}H_u\delta u\,dt$$

since (u^*,x^*) is optimal solution which satisfies the first-order necessary conditions, the coefficients of $\delta^2 u$, $\delta^2 x$, and $\delta^2 x_f$ vanish, and

$$\delta^2 J(u^*,\delta u)=\frac{1}{2}\delta x_f^T\phi_{x_f x_f}\delta x_f+\frac{1}{2}\int_{t_0}^{t_f}2W(t,\delta x,\delta x_\tau,\delta u)\,dt$$

that is

$$\delta^2 J(u^*,\eta)=\frac{1}{2}\zeta^T(t_f)\,\phi_{x_f x_f}\zeta(t_f)+\frac{1}{2}\int_{t_0}^{t_f}2W(t,\zeta,\zeta_\tau,\eta)\,dt \tag{1.9}$$

The above derivatives, evaluated along (u^*,x^*), are functions only of time, and are continuous. Since u^* minimize $J(u)$, the second variation must be positive or zero for all η, ζ satisfying the following variational conditions

$$\dot{\zeta}=f_x\zeta+f_{x_\tau}\zeta_\tau+f_u\eta \tag{1.10}$$

$$\zeta(t)=0, \qquad t_0-\tau\le t\le t_0 \tag{1.11}$$

$$\psi_{x_f}\zeta(t_f)=0, \tag{1.12}$$

where the derivatives are evaluated along (u^*,x^*).

The accessary minimum problem is the problem of finding a control variation η which minimizes (1.9) subject to (1.10)-(1.12).

Define

$$K(t,\eta,\xi,\xi_\tau,\zeta)=W(t,\eta,\xi,\xi_\tau)+\zeta^T(f_x\xi+f_{x_\tau}\xi_\tau+f_u\eta) \tag{1.13}$$

$$\theta(\xi(t_f),e)=\frac{1}{2}\xi^T(t_f)\phi_{x_f x_f}\xi(t_f)+e^T\psi_{x_f}\xi(t_f) \tag{1.14}$$

where ζ is an n-vector function of t, and e is a constant p-vector, Then the following identities hold because of the quadratic form of W.

$$2K=K_\xi\xi+K_{\xi_\tau}\xi_\tau+K_\eta\eta+\zeta^T(f_x\xi+f_{x_\tau}\xi_\tau+f_u\eta)$$

$$2\theta=\theta_{\xi(t_f)}\xi(t_f)+e^T\psi_{x_f}\xi(t_f)$$

Application of the first order necessary conditions to the accessary problem yields the accessary equations (Jacobi):

$$K_\eta=0$$

i.e., $$H_{ux}\xi+H_{ux_\tau}\xi_\tau+H_{uu}\eta+f_u^T\zeta=0, \qquad t_o\le t\le t_f, \tag{1.15}$$

$$\dot{\zeta}=-K_\xi-K_{\xi_\tau}(t+\tau)$$

i.e., $$\dot{\zeta}=-H_{xx}\xi-H_{xx_\tau}\xi_\tau-H_{xu}\eta-f_x^T\zeta-[H_{x_\tau x}\xi+H_{x_\tau x_\tau}\xi_\tau+H_{x_\tau u}\eta+f_{x_\tau}\zeta]\Big|_{t+\tau} \quad t_o\le t\le t_f-\tau, \tag{1.16}$$

$$\dot{\zeta}=-K_\xi$$

i.e., $$\dot{\zeta}=-H_{xx}\xi-H_{xx_\tau}\xi_\tau-H_{xu}\eta-f_x^T\zeta, \qquad t_f-\tau\le t\le t_f, \tag{1.17}$$

$$\zeta_f=\theta_{\xi(t_f)}$$

i.e. $$\zeta(t_f)=\phi_{x_f x_f}\xi(t_f)+\psi_{x_f}^T e \tag{1.18}$$

in addition to (1.10)-(1.12).

4. Sufficient Condition I (under hypothesis $\mathcal{H}$)

Now, we allow H_{uu} to be singular along $\hat{z}$=(x,u) and let H^+ denote the

Moore-Penrose inverse of H_{uu}. (c.f.[11]) Let $M^{+}_{n\times m}$ denote the Moore-Penrose inverse of the matrix $M_{m\times n}$. By definition, we have

$$M^{+}M\,M^{+} = M^{+}$$

$$M\,M^{+}M = M$$

$$(M\,M^{+})^{T} = M\,M^{+}$$

$$(M^{+}M)^{T} = M^{+}M$$

For any matrix M, M^{+} satisfies the following lemma,

Lemma: 1° rank M^{+} = rank M

2° $(M^{+})^{+} = M$

3° $(M^{T})^{+} = (M^{+})^{T}$

then if M symmetric => M^{+} symmetric.

4° $(M^{T}M)^{+} = M^{+}(M^{+})^{T}$

then if M non-negativity => M^{+} non-negativity.

5° $M^{+} = (M^{T}M)^{+}M^{T} = M^{T}(M\,M^{T})^{+}$

then we have $R(M^{T}) = R(M^{+})$.

The proofs of this Lemma can be found in [11].

If v is any vector in E^{m}, than $v=\hat{v}+\tilde{v}$, where $\hat{v}$ is the projection of v on $R(H_{uu})$, $\tilde{v}$ is the projection of v on $N(H_{uu})$ and $H^{+}H_{uu}v=\hat{v}$.

An extremal arc $\mathcal{E}$ is an admissible trajectory along which there exists a multiplier vector p(t) which satisfies the Euler-Laqrange equations

$$\dot{p}^T = -H_x - H_{x_\tau}(t+\tau) \qquad t_0 \le t \le t_f - \tau$$

$$\dot{p}^T = -H_x \qquad t_f - \tau \le t \le t_f$$

$$H_u = 0 \qquad t_0 \le t \le t_f$$

and $\quad p^T(t_f) = \phi_{x_f}$

We shall make use of the following hypothesis which we denote by $\mathcal{H}$ (c.f. [4]).

$\mathcal{H}$: There exists an nxn continuous symmetric matrix function $P = P(t)$ defined on $[t_0,\ t_f]$ and having piecewise continuous first and second derivatives (in fact, we only need continuously differentiable property) such that

$$N(H_{xu} + Pf_u) \supseteq N(H_{uu})$$

along $\mathcal{E}$, where $\mathcal{E}$ is an extremal arc.

(R(M) is the range of matrix M and N(M) is nullity)

Let $\mathcal{E}$ be an extremal arc without corners, let $\xi, \xi_\tau, \eta, \zeta$ be the solution of accessary equations (accessary extremal variations), that is

$$H_{ux}\xi + H_{ux_\tau}\xi_\tau + H_{uu}\eta + f_u^T\zeta = 0, \qquad t_0 \le t \le t_f \tag{1.15}$$

$$\dot{\zeta} = -H_{xx}\xi - H_{xx_\tau}\xi_\tau - H_{xu}\eta - f_x^T\zeta - [H_{x_\tau x}\xi + H_{x_\tau x_\tau}\xi_\tau + H_{x_\tau u}\eta + f_{x_\tau}^T\zeta]\Big|_{t+\tau}, \qquad t_0 \le t \le t_f - \tau \tag{1.16}$$

$$\dot{\zeta} = -H_{xx}\xi - H_{xx_\tau}\xi_\tau - H_{xu}\eta - f_x^T\zeta, \qquad t_f - \tau \le t \le t_f \tag{1.17}$$

(Jacobi equations)

By premultiplying equation (1.15) by H^+, we obtain

$$\hat{\eta}=-H^{+}(H_{ux}\xi+H_{ux_\tau}\xi_\tau+f_u^T\zeta) \qquad (c)$$

Substitution of (c) into (1.10), we obtain

$$\begin{aligned}\dot{\xi}&=f_x\xi+f_{x_\tau}\xi_\tau+f_u\hat{\eta}+f_u\tilde{\eta}\\&=f_x\xi+f_{x_\tau}\xi_\tau+f_u\tilde{\eta}-f_uH^{+}(H_{ux}\xi+H_{ux_\tau}\xi_\tau+f_u^T\zeta)\\&=(f_x-f_uH^{+}H_{ux})\xi+(f_{x_\tau}-f_uH^{+}H_{ux_\tau})\xi_\tau-f_uH^{+}f_u^T\zeta+f_u\tilde{\eta}, \qquad (d)\end{aligned}$$

Substitution of (c) into (1.17), we obtain

$$\begin{aligned}\dot{\zeta}&=-H_{xx}\xi-H_{xx_\tau}\xi_\tau-H_{xu}(\tilde{\eta}+\hat{\eta})-f_x^T\zeta, \qquad t_f-\tau\le t\le t_f\\&=-H_{xx}\xi-H_{xx_\tau}\xi_\tau-H_{xu}\tilde{\eta}+H_{xu}H^{+}(H_{ux}\xi+H_{ux_\tau}\xi_\tau+f_u^T\zeta)-f_x^T\zeta\\&=(H_{xu}H^{+}H_{ux}-H_{xx})\xi+(H_{xu}H^{+}H_{ux_\tau}-H_{xx_\tau})\xi_\tau+(H_{xu}H^{+}f_u^T-f_x^T)\zeta-H_{xu}\tilde{\eta} \qquad (e)\end{aligned}$$

Substitution of (c) into (1.16)

$$\begin{aligned}\dot{\zeta}&=-H_{xx}\xi-H_{xx_\tau}\xi_\tau-H_{xu}\eta-f_x^T\zeta-[H_{x_\tau x}\xi+H_{x_\tau x_\tau}\xi_\tau+H_{x_\tau u}\eta+f_{x_\tau}^T\zeta]\big|_{t+\tau}\\&=-H_{xx}\xi-H_{xx_\tau}\xi_\tau-H_{xu}(\tilde{\eta}+\hat{\eta})-f_x^T\zeta-[H_{x_\tau x}\xi+H_{x_\tau x_\tau}\xi_\tau+H_{x_\tau u}(\tilde{\eta}+\hat{\eta})+f_{x_\tau}^T\zeta]\big|_{t+\tau}\\&=-H_{xx}\xi-H_{xx_\tau}\xi_\tau-H_{xu}\tilde{\eta}+H_{xu}H^{+}(H_{ux}\xi+H_{ux_\tau}\xi_\tau+f_u^T\zeta)-f_x^T\zeta\\&\quad-[H_{x_\tau x}\xi+H_{x_\tau x_\tau}\xi_\tau+H_{x_\tau u}\tilde{\eta}-H_{x_\tau u}H^{+}(H_{ux}\xi+H_{ux_\tau}\xi_\tau+f_u^T\zeta)+f_{x_\tau}^T\zeta]\big|_{t+\tau}\\&=(H_{xu}H^{+}H_{ux}-H_{xx})\xi+(H_{xu}H^{+}H_{ux_\tau}-H_{xx_\tau})\xi_\tau+(H_{xu}H^{+}f_u^T-f_x^T)\zeta-H_{xu}\tilde{\eta}\\&\quad-[(H_{x_\tau x}-H_{x_\tau u}H^{+}H_{ux})\xi+(H_{x_\tau x_\tau}-H_{x_\tau u}H^{+}H_{ux_\tau})\xi_\tau+(f_{x_\tau}^T-H_{x_\tau u}H^{+}f_u^T)\zeta+H_{x_\tau u}\tilde{\eta}]\big|_{t+\tau}\\&\qquad\qquad t_0\le t\le t_f-\tau, \qquad (f)\end{aligned}$$

let

$$(g)\quad\begin{cases}A=f_x-f_uH^{+}H_{ux}, & B=f_uH^{+}f_u^T\\A_\tau=f_{x_\tau}-f_uH^{+}H_{ux_\tau}, & C=H_{xx}-H_{xu}H^{+}H_{ux}\\C_\tau=H_{xx_\tau}-H_{xu}H^{+}H_{ux_\tau}, & D=H_{x_\tau x_\tau}-H_{x_\tau u}H^{+}H_{ux_\tau}\end{cases}$$

then, $\dot{\xi}=A\xi+A_\tau\xi_\tau-B\zeta+f_u\tilde{\eta}$ (1.19)

$$\dot{\zeta}=-C\xi-C_\tau\xi_\tau-A^T\zeta-H_{xu}\tilde{\eta}-[C_\tau^T\xi+D\xi_\tau+A_\tau^T\zeta+H_{x_\tau u}\tilde{\eta}]\big|_{t+\tau}\quad t_0\le t\le t_f-\tau \qquad (1.20)$$

$$\dot{\zeta}=-C\xi-C_\tau\xi_\tau-A^T\zeta-H_{xu}\tilde{\eta}\,,\qquad t_f-\tau\le t\le t_f \qquad (1.21)$$

If we add to the integrand of $\delta^2 J$ the term $\frac{d}{dt}(\xi^T S\xi)$ where S is an arbitrary continuously differentiable symmetric matric function of t, by substituting (1.19), (1.20), (1.21) into (1.9), we obtain

$$\delta^2 J(u,\eta)=\frac{1}{2}\int_{t_0}^{t_f}[2W+\frac{d}{dt}(\xi^T S\xi)\,dt+\frac{1}{2}[\xi^T(t_f)\phi_{x_f x_f}\xi(t_f)-\xi^T(t_f)S(t_f)\xi(t_f)]$$

$$2W+\frac{d}{dt}(\xi^T S\xi)=2W+\dot{\xi}^T S\xi+\xi^T\dot{S}\xi+\xi S\dot{\xi}$$

$$=2W+(A\xi+A_\tau\xi_\tau-B\zeta+f_u\tilde{\eta})^T S\xi+\xi^T\dot{S}\xi+\xi^T S(A\xi+A_\tau\xi_\tau-B\zeta+f_u\tilde{\eta})$$

$$=\xi^T H_{xx}\xi+2\xi^T H_{xx_\tau}\xi_\tau+\xi_\tau^T H_{x_\tau x_\tau}\xi_\tau+2\xi^T H_{xu}\eta+2\xi_\tau^T H_{x_\tau u}\eta+\eta^T H_{uu}\eta+\xi^T A^T S\xi+\xi_\tau^T A_\tau^T S\xi$$

$$-\zeta^T B^T S\xi+\tilde{\eta}^T f_u^T S\xi+\xi^T\dot{S}\xi+\xi^T SA\xi+\xi^T SA_\tau\xi_\tau-\xi^T SB\zeta+\xi^T Sf\tilde{\eta}$$

To calculate $2\xi^T H_{xu}\eta$, we have,

$$2\xi^T H_{xu}\eta=2\xi^T H_{xu}(\hat{\eta}+\tilde{\eta})=2\xi^T H_{xu}\hat{\eta}+2\xi^T H_{xu}\tilde{\eta}=-2\xi^T H_{xu}H^+(H_{ux}\xi+H_{ux_\tau}\xi_\tau+f_u^T\zeta)+2\xi^T H_{xu}\tilde{\eta}$$

$$=-2\xi^T H_{xu}H^+H_{ux}\xi-2\xi^T H_{xu}H^+H_{ux_\tau}\xi_\tau-2\xi^T H_{xu}H^+f_u^T\zeta+2\xi^T H_{xu}\tilde{\eta}$$

Similarly: $2\xi_\tau^T H_{x_\tau u}\eta=2\xi_\tau^T H_{x_\tau u}(\hat{\eta}+\tilde{\eta})$

$$=-2\xi_\tau^T H_{x_\tau u}H^+H_{ux}\xi-2\xi_\tau^T H_{x_\tau u}H^+H_{ux_\tau}\xi_\tau-2\xi_\tau^T H_{x_\tau u}H^+f_u^T\zeta+2\xi_\tau^T H_{x_\tau u}\tilde{\eta}$$

To calculate $\eta^T H_{uu}\eta$,

$$\eta^T H_{uu}\eta=(\hat{\eta}+\tilde{\eta})H_{uu}(\hat{\eta}+\tilde{\eta})=\hat{\eta}^T H_{uu}\hat{\eta}=[H^+(H_{ux}\xi+H_{ux_\tau}\xi_\tau+f_u^T\zeta)]^T H_{uu}H^+(H_{ux}\xi+H_{ux_\tau}\xi_\tau+f_u^T\zeta)$$

$$=\xi^T H_{xu}H^+H_{ux}\xi+2\xi^T H_{xu}H^+H_{ux_\tau}\xi_\tau+2\xi^T H_{xu}H^+f_u^T\zeta+\xi_\tau^T H_{x_\tau u}H^+H_{ux_\tau}\xi_\tau+2\xi_\tau H_{x_\tau u}H^+f_u^T\zeta+\zeta^T f_u H^+f_u^T\zeta$$

(note: $H^+H_{uu}H^+=H^+$)

That is:

$$2W+\frac{d}{dt}(\xi^T S\xi)=\xi^T H_{xx}\xi+2\xi^T H_{xx_\tau}\xi_\tau+\xi_\tau^T H_{x_\tau x_\tau}\xi_\tau-2\xi^T H_{xu}H^\dagger H_{ux}\xi-2\xi^T H_{xu}H^\dagger H_{ux_\tau}\xi_\tau-2\xi^T H_{xu}H^\dagger f_u^T\zeta$$

$$+2\xi^T H_{xu}\tilde{\eta}-2\xi_\tau^T H_{x_\tau u}H^\dagger H_{ux}\xi-2\xi_\tau^T H_{x_\tau u}H^\dagger H_{ux_\tau}\xi_\tau-2\xi_\tau^T H_{xu}H^\dagger f_u^T\zeta+2\xi_\tau^T H_{x_\tau u}\tilde{\eta}+\xi^T H_{xu}H^\dagger H_{ux}\xi+$$

$$2\xi_\tau^T H_{x_\tau u}H^\dagger H_{ux}\xi+2\zeta^T f_u H^\dagger H_{ux}\xi+\xi_\tau^T H_{x_\tau u}H^\dagger H_{ux_\tau}\xi_\tau+2\zeta^T f_u H^\dagger H_{ux_\tau}\xi_\tau+\zeta^T f_u H^\dagger f_u^T\zeta+\xi^T A^T S\xi+\xi_\tau^T A_\tau^T S\xi$$

$$-\zeta^T B^T S\xi+\tilde{\eta}^T f_u^T S\xi+\xi^T\dot{S}\xi+\xi^T SA\xi+\xi^T SA_\tau\xi_\tau-\xi^T SB\zeta+\xi^T Sf_u\tilde{\eta}$$

Collecting terms, and using (g), we obtain

$$2W+\frac{d}{dt}(\xi^T S\xi)=\xi^T(\dot{S}+A^T S+SA+C)\xi+2\xi^T(SA_\tau+C_\tau)\xi_\tau+\xi_\tau^T D\xi_\tau-2\xi BS\zeta+2\xi^T(H_{xu}+Sf_u)\tilde{\eta}$$

$$+2\xi_\tau^T H_{x_\tau u}\tilde{\eta}+\zeta^T B\zeta$$

$$=\xi^T(\dot{S}+SA+A^T S-SBS+C)\xi+(\zeta-S\xi)^T B(\zeta-S\xi)+2\xi^T(H_{xu}+Sf_u)\tilde{\eta}$$

$$+2\xi^T(SA_\tau+C_\tau)\xi_\tau+\xi_\tau^T D\xi_\tau+2\xi_\tau^T H_{x_\tau u}\tilde{\eta}$$

let $S^*=\dot{S}+SA+A^T S-SBS+C$, and $\delta=\zeta-S\xi$

thus,

$$2W+\frac{d}{dt}(\xi^T S\xi)=\xi^T S^*\xi+\delta^T B\delta+2\xi^T(H_{xu}+Sf_u)\tilde{\eta}+2\xi^T(SA_\tau+C_\tau)\xi_\tau+\xi_\tau^T D\xi_\tau+2\xi_\tau^T H_{x_\tau u}\tilde{\eta}$$

If hypothesis #2 holds, let us choose $S=P$

then $$\delta^2 J=\frac{1}{2}\int_{t_0}^{t_f}\{\xi^T P^*\xi+\delta^T B\delta+2\xi^T(PA_\tau+C_\tau)\xi_\tau+\xi_\tau^T D\xi_\tau+2\xi_\tau^T H_{x_\tau u}\tilde{\eta}\}\,dt$$

$$+\frac{1}{2}\xi^T(t_f)(\Phi_{x_f x_f}-P(t_f))\xi(t_f) \qquad \text{(h)}$$

If we suppose that $N(H_{x_\tau u})\geq N(H_{uu})$, $(R(H_{uu})\geq R(H_{ux_\tau}))$, then (h) becomes

$$\delta^2 J=\frac{1}{2}\int_{t_0}^{t_f}\{\xi^T P^*\xi+\delta^T B\delta+2\xi^T(PA_\tau+C_\tau)\xi_\tau+\xi_\tau^T D\xi_\tau\}\,dt+\frac{1}{2}\xi^T(t_f)[\Phi_{x_f x_f}-P(t_f)]\xi(t_f) \qquad \text{(i)}$$

from (i), let $M=PA_\tau+C_\tau$,

we can rewrite $\delta^2 J$ as

$$\delta^2 J=\frac{1}{\alpha}\int_{t_0}^{t_f}[(\xi^T,\xi_\tau^T)\begin{bmatrix}P^* & M^T\\ M & D\end{bmatrix}\begin{bmatrix}\xi\\ \xi_\tau\end{bmatrix}+\delta^T B\delta]\,dt+\frac{1}{\alpha}\xi^T(t_f)[\phi_{x_f x_f}-P(t_f)]\xi(t_f)$$

let $\alpha^T=(\xi,\xi_\tau)^T$, $R=\begin{bmatrix}P^* & M^T\\ M & D\end{bmatrix}$, a symmetric matrix,

then we have

$$\delta^2 J=\frac{1}{\alpha}\int_{t_0}^{t_f}(\alpha^T R\alpha+\delta^T B\delta)\,dt+\frac{1}{\alpha}\xi^T(t_f)[\phi_{x_f x_f}-P(t_f)]\xi(t_f) \tag{1.22}$$

we have already known that the variations satisfy the equation (1.15)

If we add $f_u^T P\xi$ to both sides of equation (1.15), we obtain

$$H_{ux}\xi+H_{ux\tau}\xi_\tau+H_{uu}\eta+f_u^T P\xi=f_u^T P\xi-f_u^T\zeta$$

$$H_{uu}\eta+(H_{ux}+f_u^T P)\xi+H_{ux\tau}\xi_\tau=-f_u^T\delta$$

then, we have proved the following theorem:

Theorem 1.1 If $\mathscr{C}$ is an extremmal arc along which hypothesis $\mathscr{H}$ holds and if $N(H_{xu})\supseteq N(H_{uu})$, ξ,ζ,ξ_τ are accessary extremal variations, then

$$\delta^2 J=\frac{1}{\alpha}\int_{t_0}^{t_f}(\alpha^T R\alpha+\delta^T B\delta)\,dt+\frac{1}{\alpha}\xi^T(t_f)[\phi_{x_f x_f}-P(t_f)]\xi(t_f)$$

where $\delta=\zeta-P\xi$, $B=f_u H^+ f_u$, α, R defined as before. Furthermore,

$f_u^T\delta\in R(H_{uu})$.

Corollary: Along an extremal arc $\mathscr{C}$, suppose that H_{uu} is non-negative definite (so does H^+), hypothesis $\mathscr{H}$ holds and $N(H_{x_\tau u})\supseteq N(H_{uu})$. If $\alpha^T R\alpha\geq 0$ for all extremal variations, and also $\phi_{x_f x_f}\geq P(t_f)$, then $\delta^2 J$ is non-negative.

5. Sufficient Condition II (general case)

Note that for any nxn, symmetric, continuously differentiable, matrix function of time P(t), we have that

$$\int_{t_0}^{t_f}[\xi^T P(f_x\xi+f_{x\tau}\xi_\tau+f_u\eta-\dot{\xi})]\,dt=0$$

Now adding this identically zero integral to $\delta^2 J$ yields

$$\delta^2 J(u,\eta)=\frac{1}{\alpha}\int_{t_0}^{t_f}[2W+\xi^T Pf_x\xi+\xi^T Pf_{x\tau}\xi_\tau+\xi^T Pf_u\eta-\xi^T P\dot{\xi}]\,dt+\frac{1}{\alpha}\xi^T(t_f)\phi_{x_f x_f}\xi(t_f), \qquad (j)$$

(W as before)

In view of our assumptions on u(t) and P(t), we can integrate the term $\xi^T P\dot{\xi}$ by parts to obtain, (c.f.[5])

$$\begin{aligned}\int_{t_0}^{t_f}\xi^T P\dot{\xi}\,dt&=\int_{t_0}^{t_f}\xi^T P\,d\xi=\xi^T P\xi\Big|_{t_0}^{t_f}-\int_{t_0}^{t_f}\xi\,d\xi^T P\\&=\xi^T(t_f)P(t_f)\xi(t_f)-\int_{t_0}^{t_f}[\xi^T(\dot{P}\xi+P\dot{\xi})]\,dt\\&=\xi^T(t_f)P(t_f)\xi(t_f)-\int_{t_0}^{t_f}\xi^T\dot{P}\xi\,dt-\int_{t_0}^{t_f}\xi^T P(f_x\xi+f_{x\tau}\xi_\tau+f_u\eta)\,dt \qquad (k)\end{aligned}$$

substituting (k) into (j),

$$\begin{aligned}\delta^2 J&=\frac{1}{\alpha}\int_{t_0}^{t_f}[2W+2\xi^T Pf_x\xi+2\xi^T Pf_{x\tau}\xi_\tau+2\xi^T Pf_u\eta+\xi^T\dot{P}\xi]\,dt+\frac{1}{\alpha}\xi^T(t_f)[\phi_{x_f x_f}-P(t_f)]\xi(t_f)\\&=\frac{1}{\alpha}\int_{t_0}^{t_f}[\xi^T H_{xx}\xi+2\xi^T H_{xx\tau}\xi_\tau+\xi_\tau^T H_{x\tau x\tau}\xi_\tau+2\xi^T H_{xu}\eta+2\xi_\tau^T H_{x\tau u}\eta+\eta^T H_{uu}\eta+2\xi^T Pf_x\xi+2\xi^T Pf_{x\tau}\xi_\tau\\&\quad+2\xi^T Pf_u\eta+\xi^T\dot{P}\xi]\,dt+\frac{1}{\alpha}\xi^T(t_f)[\phi_{x_f x_f}-P(t_f)]\xi(t_f)\\&=\frac{1}{\alpha}\int_{t_0}^{t_f}(\xi^T,\xi_\tau^T,\eta^T)\begin{pmatrix}H_{xx}+Pf_x+f_x^T P+\dot{P} & H_{xx\tau}+Pf_{x\tau} & H_{xu}+Pf_u\\ H_{x\tau x}+f_{x\tau}^T P & H_{x\tau x\tau} & H_{x\tau u}\\ H_{ux}+f_u^T P & H_{ux\tau} & H_{uu}\end{pmatrix}\begin{pmatrix}\xi\\ \xi_\tau\\ \eta\end{pmatrix}dt\\&\quad+\frac{1}{\alpha}\xi^T(t_f)[\phi_{x_f x_f}-P(t_f)]\xi(t_f) \qquad (1.23)\end{aligned}$$

then, we obtain the following theorem:

Theorem 1.2 : Along an extremal arc $\mathcal{E}$, a sufficient condition for non-negative of $\delta^2 J$ is that there exists, for all t in $[t_0, t_1]$, a continuously differentiable, symmetric, matrix function of time P(t) such that

$$\begin{pmatrix} H_{xx}+Pf_x+f_x^TP+\dot{P} & H_{xx_\tau}+Pf_{x_\tau} & H_{xu}+Pf_u \\ H_{x_\tau x}+f_{x_\tau}^TP & H_{x_\tau x_\tau} & H_{x_\tau u} \\ H_{ux}+f_u^TP & H_{ux_\tau} & H_{uu} \end{pmatrix} \geq 0$$

for all t in $[t_0, t_f]$ and $\Phi_{x_f x_f} \geq P(t_f)$.

6. Sufficient Condition III (for the totally singular case)

$\delta^2 J(\eta)$ is said to be totally singular if $H_{uu}=0$ for all t in $[t_0, t_f]$. (c.f.[5])

We need the following assumption:

$\frac{\partial}{\partial\eta}\dot{M}_\eta=0$ and $-\frac{\partial}{\partial\eta}\ddot{M}_\eta>0$, for all t in $[t_0, t_f]$, where $M=W+\zeta^T(f_x\xi+f_{x_\tau}\xi_\tau+f_u\eta)$

W is defined as before, and $\xi, \xi_\tau, \eta, \zeta$ satisfies Jacobi equations.

let $F=H_{x_\tau x}\xi+H_{x_\tau x_\tau}\xi_\tau+H_{x_\tau u}\eta+f_{x_\tau}\zeta$, and suppose that $H_{uu}=0$ and $H_{x_\tau u}=0$ for all t in $[t_0, t_f]$, and we calculate $\frac{\partial}{\partial\eta}\dot{M}_\eta$ and $\frac{\partial}{\partial\eta}\ddot{M}_\eta$ under this condition.

i) calculating $\frac{\partial}{\partial\eta}M_\eta$:

$$M_\eta=\xi^TH_{xu}+\zeta^Tf_u$$

$$\frac{d}{dt}M_\eta=\dot{\xi}^TH_{xu}+\xi^T\dot{H}_{xu}+\dot{\zeta}^Tf_u+\zeta^T\dot{f}_u$$

$$=(\xi^Tf_x^T+\xi_\tau^Tf_{x_\tau}^T+\eta^Tf_u^T)H_{xu}+\xi^T\dot{H}_{xu}+\zeta^T\dot{f}_u$$

$$+(-H_{xx}\xi-H_{xx_\tau}\xi_\tau-H_{xu}\eta-f_x^T\zeta-[F]\big|_{t+\tau})^Tf_u \qquad (t_0\leq t\leq t_f-\tau)$$

$$+(-H_{xx}\xi - H_{xx\tau}\xi_\tau - H_{xu}\eta - f_x^T\zeta)^T f_u \qquad (t_f-\tau \le t \le t_f)$$

$$\frac{\partial}{\partial\eta}\dot{M}_\eta = f_u^T H_{xu} - H_{ux} f_u \qquad (l)$$

if $\frac{\partial}{\partial\eta}\dot{M}_\eta = 0 \Rightarrow f_u^T H_{xu}$ symmetric.

ii) calculating $\frac{\partial}{\partial\eta}\ddot{M}_\eta$:

$$\ddot{M}_\eta = \dot{\xi}^T f_x^T H_{xu} + \xi^T \dot{f}_x^T H_{xu} + \xi^T f_x^T \dot{H}_{xu} + \dot{\xi}_\tau^T f_{x\tau}^T H_{xu} + \xi_\tau^T \dot{f}_{x\tau}^T H_{xu} + \xi_\tau^T f_{x\tau}^T \dot{H}_{xu} + \dot{\eta}^T f_u^T H_{xu} + \eta^T \dot{f}_u^T H_{xu} + \eta^T f_u^T \dot{H}_{xu} +$$

$$\dot{\xi}^T \dot{H}_{xu} + \xi^T \ddot{H}_{xu} + \ddot{\zeta}^T f_u + \dot{\zeta}^T \dot{f}_u + \dot{\zeta}^T \dot{f}_u + \zeta^T \ddot{f}_u$$

$$= (\xi^T f_x^T + \xi_\tau^T f_{x\tau}^T + \eta^T f_u^T) f_x^T H_{xu} + \xi^T \dot{f}_x^T H_{xu} + \xi^T f_x^T \dot{H}_{xu} + \dot{\xi}_\tau^T f_{x\tau}^T H_{xu} + \xi_\tau^T \dot{f}_{x\tau}^T H_{xu} + \xi_\tau^T f_{x\tau}^T \dot{H}_{xu} + \dot{\eta}^T f_u^T H_{xu} +$$

$$\eta^T \dot{f}_u^T H_{xu} + \eta^T f_u^T \dot{H}_{xu} + (\xi^T f_x^T + \xi_\tau^T f_{x\tau}^T + \eta^T f_u^T)\dot{H}_{xu} + \xi^T \ddot{H}_{xu} + \zeta^T \ddot{f}_u$$

$$+\begin{Bmatrix} -\dot{H}_{xx}\xi - H_{xx}\dot{\xi} - \dot{H}_{xx\tau}\xi_\tau - H_{xx\tau}\dot{\xi}_\tau - \dot{H}_{xu}\eta - H_{xu}\dot{\eta} - \dot{f}_x^T\zeta - f_x^T\dot{\zeta} - \dot{F}|_{t+\tau} \\ -\dot{H}_{xx}\xi - H_{xx}\dot{\xi} - \dot{H}_{xx\tau}\xi_\tau - H_{xx\tau}\dot{\xi}_\tau - \dot{H}_{xu}\eta - H_{xu}\dot{\eta} - \dot{f}_x^T\zeta - f_x^T\dot{\zeta} \end{Bmatrix}^T f_u$$

$$+2\begin{Bmatrix} -H_{xx}\xi - H_{xx\tau}\xi_\tau - H_{xu}\eta - f_x^T\zeta - F|_{t+\tau} \\ -H_{xx}\xi - H_{xx\tau}\xi_\tau - H_{xu}\eta - f_x^T\zeta \end{Bmatrix}^T \dot{f}_u$$

$$\frac{\partial}{\partial\eta}\ddot{M}_\eta = -f_u^T H_{xx} f_u + \dot{H}_{ux} f_u + H_{ux} f_x f_u - \dot{f}_u^T H_{xu} + f_u^T f_x^T H_{xu} \qquad (m)$$

<u>Lemma 1</u>: $\frac{\partial}{\partial\eta}\ddot{M}_\eta = -f_u^T H_{xx} f_u + \dot{H}_{ux} f_u + H_{ux} f_x f_u - \dot{f}_u^T H_{xu} + f_u^T f_x^T H_{xu}$.

(by using (l), we can proof <u>Lemma 1</u>)

<u>Lemma 2</u>: Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be symmetric matrix, if A_{22} is positive, and suppose that $A_{11} - A_{12}A_{22}^{-1}A_{21} \ge 0$, then A is non-negative.

Proof: take $\begin{pmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{pmatrix}$, then we get,

$$\begin{pmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ -A_{12}A_{22}^{-1} & I \end{pmatrix} = \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{pmatrix} \geq 0$$

$$\Rightarrow \quad \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{is non-negative.}$$

Thus, for the totally singular case, and when the assumption is satisfied, we obtain the following theorem.

Theorem 1.3 Along an extremal arc $\mathcal{E}$, suppose that $H_{ux_\tau}=0$, for t in $[t_o, t_f]$. $H_{x_\tau x_\tau}$ is positive and if there exists a function S(t) which satisfies, for all t in $[t_o, t_f]$, the matrix equations and inequities,

$$-\dot{S}=H_{xx}+Sf_x+f_x^TS+[(f_x f_u-\dot{f}_u)^TS+f_u^TH_{xx}-H_{ux}f_x-\dot{H}_{ux}]\,[\tfrac{\partial}{\partial\eta}\ddot{M}_\eta]^{-1}[(f_x f_u-\dot{f}_u)^TS+f_u^TH_{xx}-H_{ux}f_x-\dot{H}_{ux}]$$

$$(1.24)$$

$$\dot{S}+H_{xx}+Sf_x+f_x^TS \geq (H_{x_\tau x}+f_{x_\tau}^TS)\,H_{x_\tau x_\tau}^{-1}(H_{x_\tau x}+f_{x_\tau}^TS) \qquad (1.25)$$

$$H_{ux}(t_f)+f_u^T(t_f)S(t_f)=0 \qquad (1.26)$$

$$\phi_{x_fx_f}-S(t_f) \geq 0 \qquad (1.27)$$

where $M=W+\zeta^T(f_x\xi+f_{x_\tau}\xi_\tau+f_u\eta)$, and $\xi, \xi_\tau, \eta, \zeta$ are extremal variations, then there exists a P(t) which satisfies Theorem 1.2.

Proof: from $-\frac{\partial}{\partial\eta}\ddot{M}_\eta > 0$ and (1.24), we see that

$$\dot{S}+H_{xx}+Sf_x+f_x^TS \geq 0, \quad \text{for all t in } [t_o, t_f]. \qquad (1.28)$$

Next, premultiplying (1.24) by f_u^T and using Lemma 1 and (1) we obtain,

$$-f_u^T\dot{S}=(H_{ux}+f_u^TS)f_x+\dot{H}_{ux}+\dot{f}_u^TS+[f_u^TS(f_xf_u-\dot{f}_u)+H_{ux}(f_xf_u-\dot{f}_u)]$$
$$[\tfrac{\partial}{\partial\eta}\ddot{M}_\eta]^{-1}[(f_xf_u-\dot{f}_u)^TS+f_u^TH_{xx}-H_{ux}f_x-\dot{H}_{ux}] \qquad (1.29)$$

which, re-arranged, is

$$-\frac{d}{dt}(H_{ux}+f_u^TS)=(H_{ux}+f_u^TS)\{f_x+(\ddot{f}_x f_{u_i}-\dot{f}_u)[\frac{\partial}{\partial\eta}\ddot{M}_\eta]^{-1}[f_x f_u-\dot{f}_u)S+f_u^TH_{xx}-H_{ux}f_x-\dot{H}_{ux}]] \qquad (1.30)$$

Now this is an ordinary linear homogeneous differential equation for $(H_{ux}+f_u^TS)$ which satisfies the boundary condition (1.26). Consequently,

$$H_{ux}+f_u^TS=0, \qquad \text{for all } t \text{ in } [t_0, t_f], \qquad (1.31)$$

then we have

$$\begin{pmatrix} H_{xx}+Sf_x+f_x^TS+\dot{S} & H_{xx_\tau}+Sf_{x_\tau} & H_{xu}+Sf_u \\ H_{x_\tau x}+f_{x_\tau}^TS & H_{x_\tau x_\tau} & H_{x_\tau u} \\ H_{ux}+f_u^TS & H_{ux_\tau} & H_{uu} \end{pmatrix} = \begin{pmatrix} H_{xx}+Sf_x+f_x^TS+\dot{S} & H_{xx_\tau}+Sf_{x_\tau} & 0 \\ H_{x_\tau x}+f_{x_\tau}^TS & H_{x_\tau x_\tau} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $H_{xx}+Sf_x+f_x^TS+\dot{S}\geq 0$. Since $H_{x_\tau x_\tau}$ positive, and S satisfies (1.25), then, by

<u>Lemma 2</u>, $$\begin{pmatrix} H_{xx}+Sf_x+f_x^TS+\dot{S} & H_{xx_\tau}+Sf_{x_\tau} \\ H_{x_\tau x}+f_{x_\tau}^TS & H_{x_\tau x_\tau} \end{pmatrix} \geqslant 0$$

identifying P(t) with S(t), we see that the theorem is true.

7. Sufficient Condition IV (for the non-singular case, $H_{uu}>0$.) (c.f.[5])

Considering the identically zero equatity

$$\check{\xi}^T(t_0)S(t_0)\check{\xi}(t_0)-\check{\xi}^T(t_f)S(t_f)\check{\xi}(t_f)+\int_{t_0}^{t_f}\frac{d}{dt}[\check{\xi}^T(t)S(t)\check{\xi}(t)]\,dt$$

where S(t) is a continuously differentiable symmetric matrix function defined in $[t_0, t_f]$, and calculate $\int_{t_0}^{t_f}\frac{d}{dt}[\check{\xi}^T(t)S(t)\check{\xi}(t)]\,dt$,

$$\text{which} = \int_{t_0}^{t_f}(2\dot{\check{\xi}}^TS\check{\xi}+\check{\xi}^T\dot{S}\check{\xi})\,dt$$

$$= \int_{t_0}^{t_f}[\check{\xi}^T(\dot{S}+f_x^TS+Sf_x)\check{\xi}+2\check{\xi}_\tau^Tf_{x_\tau}^TS\check{\xi}+2\eta^Tf_u^TS\check{\xi}]\,dt$$

$$= \int_{t_0}^{t_f} (\xi^T, \xi_\tau^T, \eta^T, \eta^T) \begin{pmatrix} \dot{S}+f_x^T S+Sf_x & Sf_{x_\tau} & Sf_u & 0 \\ f_{x_\tau}^T S & 0 & 0 & 0 \\ f_u^T S & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \xi_\tau \\ \eta \\ \eta \end{pmatrix} dt$$

let
$$S_1 = \begin{pmatrix} \dot{S}+f_x^T S+Sf_x & Sf_{x_\tau} \\ f_{x_\tau}^T S & 0 \end{pmatrix} \qquad S_2 = \begin{pmatrix} f_u^T S & 0 \\ 0 & 0 \end{pmatrix}$$

and $\alpha^T = (\xi^T, \xi_\tau^T)$ $\beta^T = (\eta^T, \eta^T)$

then $$\int_{t_0}^{t_f} \frac{d}{dt}(\xi^T S \xi)\, dt = \int_{t_0}^{t_f} (\alpha^T, \beta^T) \begin{pmatrix} S_1 & S_2^T \\ S_2 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} dt \qquad (1.32)$$

$$\delta^2 J(\eta) = \frac{1}{2}\int_{t_0}^{t_f} (\xi^T, \xi_\tau^T, \eta^T, \eta^T) \begin{pmatrix} H_{xx} & H_{xx_\tau} & H_{xu} & 0 \\ H_{x_\tau x} & H_{x_\tau x_\tau} & H_{x_\tau u} & 0 \\ H_{ux} & H_{ux_\tau} & \frac{1}{2}H_{uu} & 0 \\ 0 & 0 & 0 & \frac{1}{2}H_{uu} \end{pmatrix} \begin{pmatrix} \xi \\ \xi_\tau \\ \eta \\ \eta \end{pmatrix} dt + \frac{1}{2}\xi^T(t_f)\phi_{x_f x_f}\xi(t_f)$$

where $H_{uu} > 0$.

Let
$$H_1 = \begin{pmatrix} H_{xx} & H_{x_\tau x} \\ H_{x_\tau x} & H_{x_\tau x_\tau} \end{pmatrix} \qquad H_2 = \begin{pmatrix} H_{ux} & H_{ux_\tau} \\ 0 & 0 \end{pmatrix} \qquad 2R = \begin{pmatrix} H_{uu} & 0 \\ 0 & H_{uu} \end{pmatrix}$$

then $$\delta^2 J(\eta) = \frac{1}{2}\int_{t_0}^{t_f} (\alpha^T, \beta^T) \begin{pmatrix} H_1 & H_2^T \\ H_2 & R \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} dt + \frac{1}{2}\xi^T(t_f)\phi_{x_f x_f}\xi(t_f) \qquad (1.33)$$

substituting (1.32) into (1.33),

$$\delta^2 J(\eta) = \frac{1}{2}\int_{t_0}^{t_f} (\alpha^T, \beta^T) \begin{pmatrix} S_1+H_1 & (S_2+H_2)^T \\ S_2+H_2 & R \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} dt + \frac{1}{2}\xi^T(t_f)[\phi_{x_f x_f} - S(t_f)]\xi(t_f)$$

$$=\frac{1}{2}\int_{t_0}^{t_f}\{\alpha^T[(S_1+H_1)-(\dot{S}_2+H_2)^T R^{-1}(S_2+H_2)]\alpha+[\beta+R^{-1}(S_2+H_2)\alpha]^T R[\beta+R^{-1}(S_2+H_2)\alpha]\}\,dt$$

$$+\xi^T(t_f)[\phi_{x_f x_f}-S(t_f)]\xi(t_f). \tag{1.34}$$

As the term $[\beta+R^{-1}(S_2+H_2)\alpha] R[\beta+R^{-1}(S_2+H_2)\alpha] \geq 0$ $\quad (R > 0)$

so $\delta^2 J(\eta) \geq 0$ if $\quad (S_1+H_1)-(S_2+H_2)^T R^{-1}(S_2+H_2) \geq 0 \quad (1.35)$

and $\phi_{x_f x_f} \geq S(t_f) \quad (1.36)$

thus, we obtain the following theorem

Theorem 1.4: Along an extremal arc $\mathscr{E}$, suppose $H_{uu}>0$, if there exists a continuously differentiable symmetric function $S(t)$ defined in $[t_0,t_f]$, which satisfies (1.35) and (1.36), then $\delta^2 J(\eta)$ is non-negative.

Corollary: If $(S_1+H_1)-(S_2+H_2) R^{-1}(S_2+H_2) > 0$, then $\delta^2 J(\eta)$ is positive definite.

Proof: If $\delta^2 J(\eta) = 0$, $\Rightarrow$ $\alpha^T[(S_1+H_1)-(S_2+H_2) R^{-1}(S_2+H_2)]\alpha = 0$

and $[\beta+R^{-1}(S_2+H_2)\alpha]^T R[\beta+R^{-1}(S_2+H_2)\alpha] = 0$ and $\phi_{x_f x_f} = S(t_f)$

By the first equation, we have $\alpha = 0$, then by the second equation, we have $\beta = 0$, i.e. $\eta = 0$, hence, $\delta^2 J(\eta)$ is positive definite.

8. Necessary Condition for Optimal Solution (c.f.[9])

The system is represented by

$\dot{x}=f(t,x,x_\tau,u), \qquad t_0 \leq t \leq t_f$

$x=x_0(t) \qquad t_0-\tau \leq t \leq t_0$

We want to operate the system so as to minimize the cost function (1.1).

Define the Hamiltonian function $H=L+p^Tf$.

We now assume that the totally singular control u satisfies the conditions (1.5) (1.6) (1.7) (1.8). (p9)

We now seek a necessary condition for u to be an optimal solution. At first, we define

$$J(u)=\int_{t_0}^{t_f}\{H(t,x,x_\tau,u,p)-p^T\dot{x}\}\,dt+\varphi(x(t_f))$$

as before, we see that the second variation of J can be written as

$$\delta^2J(\eta)=\frac{1}{2}\xi^T(t_f)\varphi_{xx}(t_f)\xi(t_f)+\frac{1}{2}\int_{t_0}^{t_f}\{\xi^TH_{xx}\xi+\xi_\tau^TH_{x_\tau x_\tau}\xi_\tau+2\eta^TH_{ux}\xi+2\eta^TH_{ux_\tau}\xi_\tau+2\xi_\tau^TH_{x_\tau x}\xi\}\,dt \tag{1.37}$$

and that the pair (ξ,η) satisfy the following condition,

$$\dot{\xi}(t)=A(t)\xi(t)+B(t)\xi(t-\tau)+D(t)\eta(t), \qquad t\geq t_0 \tag{1.38}$$

with $\xi(t)=0$ on $[t_0-\tau,t_0]$, and where

$$A(t)=f_x(t),\quad B(t)=f_{x_\tau}(t),\quad D(t)=f_u(t).$$

The following relationship hold:

$$\int_{t_0}^{t_f}\xi_\tau^TH_{x_\tau x_\tau}\xi_\tau dt=\int_{t_0}^{t_f-\tau}\xi^TH_{x_\tau x_\tau}(t+\tau)\xi\,dt$$

$$\int_{t_0}^{t_f}\eta^TH_{ux_\tau}\xi_\tau dt=\int_{t_0}^{t_f-\tau}\eta^T(t+\tau)H_{ux_\tau}(t+\tau)\xi\,dt$$

$$\int_{t_0}^{t_f}\xi_\tau^TH_{x_\tau x}\xi\,dt=\int_{t_0}^{t_f-\tau}\xi^TH_{x_\tau x}(t+\tau)\xi(t+\tau)\,dt$$

let $P(t)$ be a symmetric differentiable nxn matrix defined on $[t_0,t_f]$, then

$$\int_{t_0}^{t_f}\frac{d}{dt}(\xi^TP\xi)\,dt=\xi^T(t_f)P(t_f)\xi(t_f)=\int_{t_0}^{t_f}\{\dot{\xi}^TP\xi+\xi^T\dot{P}\xi+\xi^TP\dot{\xi}\}\,dt$$

using (1.38), we have

$$\int_{t_0}^{t_f} \xi^T P \dot{\xi}\, dt = \xi^T(t_f)P(t_f)\xi(t_f) - \int_{t_0}^{t_f} [\xi^T A^T P \xi + \xi^T(t-\tau)B^T P \xi + \xi^T \dot{P} \xi]\, dt \tag{1.39}$$

adding the integral (=0) $\int_{t_0}^{t_f} \frac{1}{\alpha} \xi^T P [A\xi + B\xi(t-\tau) + D\eta - \dot{\xi}]\, dt$ to (1.37) and using (1.39) gives

$$\begin{aligned}\delta^2 J(\eta) = &\int_{t_0}^{t_f-\tau} \frac{1}{\alpha}\xi^T [H_{xx} + H_{x_\tau x_\tau}(t+\tau) A^T P + PA + \dot{P}]\xi\, dt + \int_{t_f-\tau}^{t_f} \frac{1}{\alpha}\xi [H_{xx} + A^T P + PA + \dot{P}]\xi\, dt \\ &+ \int_{t_0}^{t_f-\tau} \xi^T [B(t+\tau)P(t+\tau) + H_{x_\tau x}(t+\tau)]\xi(t+\tau)\, dt + \int_{t_0}^{t_f} \eta^T [H_{ux} + D^T P]\xi\, dt + \\ &+ \int_{t_0}^{t_f-\tau} \eta^T(t+\tau) H_{ux_\tau}(t+\tau)\xi\, dt + \frac{1}{\alpha}\xi^T(t_f)[\varphi_{x_f x_f} - P(t_f)]\xi(t_f)\end{aligned} \tag{1.40}$$

We now consider the special variation,

$$\eta^*(t) = \begin{cases} 0 & t < t^* \\ \beta & t^* \le t \le t^* + \varepsilon \\ 0 & t^* + \varepsilon < t \le t_f \end{cases} \tag{1.41}$$

where $\tau > \varepsilon > 0$ and β is a constant m-vector.

It can be shown that the solution of (1.38) with the given initial condition can be written as (c.f.[6])

$$\xi(t) = \int_{t_0}^{t} Y(s,t)D(s)\eta(s)\, ds \tag{1.42}$$

where the nxn matrix Y(s,t) is defined by the following equations:

$$\frac{\partial}{\partial s}Y(s,t) = -Y(s,t)A(s) - Y(s+\tau,t)B(s+\tau), \qquad t_0 \le s \le t-\tau$$

$$\frac{\partial}{\partial s}Y(s,t) = -Y(s,t)A(s), \qquad t-\tau < s \le t$$

with $Y(t,t)=I$, $Y(s,t)=0$, $s > t$.

using $\eta^*(t)$ from (1.41) substituting into (1.42) gives

$$\zeta^*(t)=\int_{t^*}^{t^*+\varepsilon} Y(s,t)D(s)\beta\, ds, \quad \text{for } t \geq t^*+\varepsilon \tag{1.43}$$

For suitably small ε, equation (1.43) gives

$$\zeta^*(t)= \varepsilon Y(t^*,t)D(t^*)\beta, \qquad t^*+\varepsilon \leq t \leq t_f \tag{1.44}$$

hence, $\zeta^*(t^*+\varepsilon)= \varepsilon Y(t^*,t^*+\varepsilon)D(t^*)\beta = \varepsilon\{Y(t^*,t^*)+[\frac{\partial Y(t^*,t^*)}{\partial t}]\varepsilon + \text{higher order terms}\} D(t^*)\beta.$

hence, $\zeta^*(t^*+\varepsilon)= \varepsilon D(t^*)\beta.$ to first order in ε. (1.45)

Assume that $\zeta^*(t)$ is linear in t over $[t^*,t^*+\varepsilon]$. Then, from (1.45) and the fact $\zeta^*(t^*)=0$, we obtain

$$\zeta^*(t)=D(t^*)\beta(t-t^*), \qquad t\in [t^*, t^*+\varepsilon] \tag{1.46}$$

Define $\Omega(t)=B^T(t+\tau)P(t+\tau)+H_{x_\tau x}(t+\tau)$.

Using (1.44) and (1.46), we can write

$$\int_{t_0}^{t_f-\tau}\zeta^{*T}(t)\Omega(t)\zeta^*(t+\tau)\, dt= \varepsilon\int_{t^*}^{t^*+\varepsilon}\beta^T D^T(t^*)\Omega(t)Y(t^*, t+\tau)D(t^*)\beta(t-t^*)dt$$
$$+ \varepsilon^2\beta^T\Delta_1(t^*+\varepsilon)\beta, \tag{1.47}$$

where $\Delta_1(t^*)=\int_{t_0}^{t_f-\tau} D^T(t^*)Y(t^*, t)\Omega(t)Y(t^*, t+\tau)D(t^*)\, dt.$

For sufficiently small ε, we can assume that $\Omega(t)$ and $Y(t^*, t+\tau)$ are constant over $[t^*, t^*+\varepsilon]$ with values $\Omega(t^*)$ and $Y(t^*, t^*+\tau)$, respectively.

Integration of the first term on the right-hand side of (1.47) then gives a term proportional to ε^3, and this can be neglected in comparison with the second term on the right-hand side of (1.47).

Again, for suitably small ε, and by continuity of $\Delta_1(\circ)$, we can write

$\Delta_1(t^*+\varepsilon)=\Delta_1(t^*)$. Hence,

$$\int_{t_0}^{t_f-\tau}\xi^{*T}(t)\Omega(t)\xi(t+\tau)\,dt=\varepsilon^2\beta^T\Delta_1(t^*)\beta \tag{1.48}$$

Also, the special variation $\eta^*(t)$ gives

$$\int_{t_0}^{t_f-\tau}\eta^{*T}(t+\tau)H_{ux_\tau}(t+\tau)\xi^*(t)\,dt=0. \tag{1.49}$$

Finally, from (1.46), we have

$$\int_{t_0}^{t_f}\eta^{*T}(t)[H_{ux}(t)+D^T(t)P(t)]\xi^*(t)\,dt$$
$$=\beta^T[H_{ux}(t^*)+D^T(t^*)P(t^*)]D(t^*)\beta\int_{t^*}^{t^*+\varepsilon}(t-t^*)\,dt=\frac{1}{2}\varepsilon^2\beta^T\Delta_2(t^*)\beta, \tag{1.50}$$

where $\quad \Delta_2(t^*)=[H_{ux}(t^*)+D^T(t^*)P(t^*)]D(t^*)$.

Since u is assumed to be optimal, it follows that $\delta^2 J \geq 0$ for all permissible variations. Hence, if we select $P(t)$ to satisfy:

$$\dot{P}(t)=-P(t)A(t)-A^T(t)P(t)-H_{xx}(t)-H_{x_\tau x_\tau}(t+\tau), \qquad t_0 \leq t \leq t_f-\tau$$

$$\dot{P}(t)=-P(t)A(t)-A^T(t)P(t)-H_{xx}(t), \qquad t_f-\tau < t \leq t_f$$

$$P(t_f)=\varphi_{xx}(t_f)$$

and using (1.48) -(1.50) in (1.40), we see that $\delta^2 J \geq 0$ implies that

$\Delta_1(\circledR)+\frac{1}{2}\Delta_2(\circledR) \geq 0, \qquad$ for $\circledR \in [t_0, t_f]$, is necessary condition. Hence we have

<u>Theorem 1.5</u>: A necessary condition for u being the optimal solution of system is that

$$\Delta_1(\circledR)+\frac{1}{2}\Delta_2(\circledR) \geq 0, \qquad \text{for } \circledR \in [t_0, t_f].$$

We can rewrite (1.40) as follows

$$\delta^2 J(\eta)=\frac{1}{2}\int_{t_0}^{t_f-\tau}\xi^T(t)[H_{xx}(t)+H_{x_\tau x_\tau}(t+\tau)+A^T(t)P(t)+P(t)A(t)+\dot{P}(t)]\xi(t)\,dt$$

$$+\frac{1}{2}\int_{t_f-\tau}^{t_f}\xi^T(t)[H_{xx}(t)+A^T(t)P(t)+P(t)A(t)+\dot{P}(t)]\xi(t)\,dt$$

$$+\frac{1}{2}\int_{t_0}^{t_f}(\xi,\xi_\tau,\eta)\begin{pmatrix} 0 & PB+H_{xx_\tau} & H_{xu}+PD^T \\ B^TP+H_{x_\tau x} & 0 & H_{x_\tau u} \\ H_{ux}+DP & H_{ux_\tau} & 0 \end{pmatrix}\begin{pmatrix}\xi \\ \xi_\tau \\ \eta\end{pmatrix}dt$$

$$+\frac{1}{2}\xi^T(t_f)[\varphi_{xx}(t_f)-P(t_f)]\xi(t_f).$$

Then we obtain the following theorem of sufficient condition for non-netativity of $\delta^2 J(\eta)$.

Theorem 1.6: If the solution of the following equation

$$\dot{P}(t)=-P(t)A(t)-A^T(t)P(t)-H_{xx}(t)-H_{x_\tau x_\tau}(t+\tau), \qquad t_0 \le t \le t_f-\tau$$

$$\dot{P}(t)=-P(t)A(t)-A^T(t)P(t)-H_{xx}(t), \qquad t_f-\tau \le t \le t_f$$

$$P(t_f)=\varphi_{xx}(t_f)$$

P(t) which satisfies

$$\begin{pmatrix} 0 & PB+H_{xx_\tau} & H_{xu}+PD^T \\ B^TP+H_{x_\tau x} & 0 & H_{x_\tau u} \\ H_{ux}+DP & H_{ux_\tau} & 0 \end{pmatrix} \ge 0$$

then $\delta^2 J(\eta)$ is non-negative.

9. Sufficient Condition V (for strongly positive definite)

Let U be the class of controls which are piecewise continuous m-vector functions of time on $[t_0, t_f]$.

$\delta^2 J(\eta)$ is said to be strongly positive definite if for each η in U and some $k > 0$,

$$\delta^2 J(\eta) \geq k\|\eta\|^2 \tag{1.51}$$

where $\|\eta\|$ is some suitable norm defined on U. (c.f.[5])

Now considering $H_{uu} > 0$,

$$\delta^2 J(\eta,\varepsilon) = \frac{1}{2}\int_{t_0}^{t_f} \{2W - \eta^T H_{uu}\eta + \frac{1}{1-\varepsilon}\eta^T H_{uu}\eta\}\, dt + \frac{1}{2}\xi^T(t_f)\,\Phi_{x_f x_f}\xi(t_f).$$

From the Corollary of Theorem 1.4, this functional is positive definite if $1-\varepsilon > 0$ and if the following inequality and equation has a solution $S(t,\varepsilon)$ defined for all t in $[t_0, t_f]$,

$$[S_1(t,\varepsilon)+H_1]-[S_2(t,\varepsilon)+H_2](1-\varepsilon)R^{-1}[S_2(t,\varepsilon)+H_2] > 0 \tag{1.52a}$$

where S_1, S_2, H_1, H_2, R defined as Theorem 1.4,

$$S(t_f,\varepsilon) = \Phi_{x_f x_f}. \tag{1.52b}$$

Now we suppose that f_x, f_{x_τ}, f_u and H_{uu}, H_{xx_τ}, H_{xx}, H_{xu}, $H_{x_\tau x_\tau}$, $H_{x_\tau u}$ are all continuous in t. We suppose there exists one-parameter continuously differentiable symmetric function $S(t,\varepsilon)$ which has the following property: $S(t,0)=S(t)$ which satisfies Theorem 1.4, and if we have that $S(t,\varepsilon)$ and its derivative $\dot{S}(t,\varepsilon)$ are continuous functions of ε at $\varepsilon=0$. Then, for suffieiently small, there exists $S(t,\varepsilon)$ which satisfies (1.52a) for all t in $[t_0, t_f]$. Therefore, for $\varepsilon < 0$ sufficiently small, $\delta^2 J(\eta,\varepsilon)$ is positive definite.

Next, we note that

$$\delta^2 J(\eta,\varepsilon)= \delta^2 J(\eta)+\frac{\varepsilon}{2(1-\varepsilon)}\int_{t_0}^{t_f}\eta^T H_{uu}\eta\, dt \geq 0$$

so that

$$\delta^2 J(\eta) \geq -\frac{\varepsilon}{2(1-\varepsilon)}\int_{t_0}^{t_f}\eta^T H_{uu}\eta\, dt \tag{1.53}$$

and because of $H_{uu} > 0$, we conclude from (1.53) that

$$\delta^2 J(\eta) \geq -\frac{\varepsilon k_1}{2(1-\varepsilon)}\int_{t_0}^{t_f}\eta^T\eta\, dt, \qquad k_1 > 0, \tag{1.54}$$

so that $\delta^2 J(\eta) \geq k\|\eta(t)\|^2$ (1.55)

where $k=-\frac{\varepsilon k_1}{2(1-\varepsilon)} > 0$ and $\|\eta(t)\| = \int_{t_0}^{t_f}\eta^T\eta\, dt$

which implies that $\delta^2 J(\eta)$ is strongly positive. Thus we obtain the following theorem,

<u>Theorem 1.7</u>: A sufficiently condition for $\delta^2 J(\eta)$ to be strongly positive is that there exists, for all t in $[t_0, t_f]$, a one-parameter continuously differentiable symmetric function $S(t,\varepsilon)$ which is a continuous function of ε at $\varepsilon=0$, so is its derivative. And $S(t,0)=S(t)$ which satisfies the following conditions,

$$(S_1+H_1)-(S_2+H_2)^T R^{-1}(S_2+H_2) > 0$$

and $S(t_f)=\phi_{x_f x_f}$, where S_1, S_2, H_1, H_2, R defined as <u>Theorem 1.4</u>.

CHAPTER 2

CONTROLLABILTY AND NORMALITY

In the Ref. [8] and [10], the definition of Controllability and Normality were defined and discussed by V. B. Hass, but she only dealt with the problems, in which there is no delay argument. In this Chapter, we extend her results to the problems with delay argument.

1. Controllability of Linear Systems

We shall consider the system described by the linear vector differential equation with delay

$$\dot{x}=A(t)x(t)+B(t)x(t-\tau)+C(t)u(t), \qquad t\in[t_0, t_f] \tag{2.1}$$

$$x(t)=x_0(t), \qquad t\in[t_0-\tau, t_0]$$

where $x_0(t)\in C([t_0-\tau, t_0]; E^n)$ and $A(t)$ and $B(t)$ are nxn matrices, $C(t)$ is an nxm matrix and the elements of these matrices are piecewise continuous functions on $[t_0, t_f]$. For the theorem in this section we could suppose that A, B are essentially bounded on $[t_0, t_f]$ and that the components of admissible controls belong to $L^2[t_0, t_f]$. We will denote the solution of (2.1) by $x=x(t;t_0, x_0(t), u)$, displaying the initial time, initial state funtion and the associated admissible control function. Let D be a constant rxn matrix with $r \le n$.

We shall say that (2.1) is controllable to the hyperplane $Dx=0$ (2.2), on the time interval $[t_0, t_f]$ if given any arbitrary initial state function $x_0(t)$, there exists an admissible control function $u(t)$, defined on $[t_0, t_f]$ and depending on $x_0(t)$ such that the trajectory of (2.1) corresponding to this control satisfies

$$Dx(t_f;\, t_0,\, x_0(t),\, u)=0$$

Theorem 2.1: If D has maximal rank and initial function $x_0(t)$ satisfies

$$D\int_{t_0-\tau}^{t_0} X(t_f,\, s+\tau)B(s+\tau)x_0(s)ds=0, \tag{2.3}$$

where $\frac{\partial}{\partial t}X(t,s)=A(t)X(t,s)+B(t)X(t-\tau,\, s)$ for $(t,s)\in[s,t_f]\times[t_0,\, t_f]$, and

$$X(s,s)=I, \qquad X(t,s)=0 \qquad \text{for } (t,s)\in[t_0-\tau,\, s)\times[t_0,\, t_f],$$

then (2.1) is controllable to the hyperplane (2.2) iff the rxr matrix,

$$V(t_0,\, t_f)=D\int_{t_0}^{t_f} X(t\ ,s)C(s)C^T(s)X^T(t_f,s)\, ds\, D^T, \tag{2.4}$$

is non-singular.

Proof: We have aleady noted that the solution of linear system (2.1) can be represented by, at time t_f,

$$x(t_f;\, t_0,\, x_0(t),u)=x(t_f,\, x_0(t))+\int_{t_0}^{t_f} X(t_f,s)C(s)u(s)\, ds. \tag{2.5}$$

and where

$$x(t_f,\, x_0(t))=X(t_f,\, t_0)x_0(t_0)+\int_{t_0-\tau}^{t_0} X(t_f,s+\tau)B(s+\tau)x_0(s)\, ds$$

where $X(t;s)$ is a unique nxn matric solution, defined on $[t_0-\tau, t_f]\times[t_0, t_f]$, of

$$\frac{\partial}{\partial t}X(t,s)=A(t)X(t,s)+B(t)X(t-\tau,s)$$

for $(t,s)\in[s, t_f]\times[t_0, t_f]$, and

$$X(s,s)=I. \qquad X(t,s)=0 \qquad \text{for } (t, s)\in[t_0-\tau, s)\times[t_0, t_f].$$

Premultiplying (2.5) by D we obtain,

$$Dx(t_f; t_0, x_0(t), u)=Dx(t_f,x_0(t))+D\int_{t_0}^{t_f}X(t_f, s)C(s)u(s)ds$$

$$=DX(t_f, t_0)x_0(t_0)+D\int_{t_0-\tau}^{t_0}X(t_f, s+\tau)B(s+\tau)x_0(s)ds+D\int_{t_0}^{t_f}X(t_f,s)C(s)u(s)ds \qquad (2.6)$$

If $V(t_0, t_f)$ is non-singular, let

$$u(t)=-C^T(t)X^T(t_f, t)D^T V^{-1}(t_0,t_f)DX(t_f, t_0)x_0(t_0)$$

then $u(t)$ is piecewise continuous on $[t_0,t_f]$ and

$$Dx(t_f; t_0, x_0(t), u)=D\int_{t_0-\tau}^{t_0}X(t_f, s+\tau)B(s+\tau)x_0(s)ds=0$$

=> (2.1) is controllable to the hyperplane (2.2).

Conversely, suppose $V(t_0, t_f)$ is singular, then there is a non-vanishing vector v in E^r such that

$$v^TD\int_{t_0}^{t_f}X(t_f, s)C(s)C^T(s)X^T(t_f,s)dsD^Tv=0$$

since D has maximal rank, then $D^Tv\neq0$

hence $C^T(t)X^T(t_f, t)D^Tv\equiv0$ on $[t_0, t_f]$ (2.7)

except for a finite number of points. If (2.1) is controllable to (2.2) then for arbitrary $x_0(t)$ there exists admisible $u(t)$, such that

$$Dx(t_f; t_0, x_0(t), u)=0$$

$$\Rightarrow \ -DX(t_f, t_0)x_0(t_0)=\int_{t_0}^{t_f} DX(t_f, s)C(s)u(s)ds.$$

Let $x_0(t_0)=X^T(t_f, t_0)D^Tv$, then $x_0(t_0)\neq 0$, and

$$-DX(t_f, t_0)X^T(t_f, t_0)D^Tv=\int_{t_0}^{t_f} DX(t_f,s)C(s)u(s)ds$$

Premultiplying this last equation by v^T and using (2.7) we find that

$$-|x_0(t_0)|^2=v^TDX(t_f, t_0)X^T(t_f, t_0)D^Tv=0.$$

This contradition proves the theorem.

2. Normality for the Problem of Bolza

If we let

$$H(t, x, x_\tau, u, p_0, p)=p_0L(t, x, x_\tau, u)+p^T(t)f(t, x, x_\tau, u)$$

then (1.8) becomes $\quad p^T(t_f)=p_0\phi_x(x(t_f))+v^T\psi_x(x(t_f)),$ (2.8)

If $p_0=0$, then equation (2.8) becomes $\quad p^T(t_f)=v^T\psi_x(x(t_f))$ (2.9)

Together with equations (1.6), (1.7) and (1.5), we have

$$\dot{p}=-f_x^Tp-f_{x_\tau}^T\,p|_{t+\tau}, \qquad t_0\le t\le t_f-\tau \tag{2.10}$$

$$\dot{p}=-f_x^Tp, \qquad t_f-\tau\le t\le t_f \tag{2.11}$$

$$p^Tf_u=0 \qquad t_0\le t\le t_f \tag{2.12}$$

We say that an extremal arc $\xi=(x^*, u^*)$ has order of abnormality k on $[t_1, t_f]\subset[t_0, t_f]$ for the problem of Bolza considered here if there are exactly k linearly independent solutions of equations (2.9), (2.10), (2.11) and (2.12) on this interval. If $k=0$, then ξ is said normal on this interval.

Let $A(t)=f_x(t)$ and $B(t)=f_{x_t}(t)$, then (2.10) and (2.11) become

$$\dot{p}(t)+A(t)^T p(t)+B(t+\tau)^T p(t+\tau)=0 \qquad t_0 \le t \le t_f-\tau$$

$$\dot{p}(t)+A(t)^T p(t)=0 \qquad t_f-\tau \le t \le t_f$$

$$p(t_f)=\psi_x^T(x^*(t_f))v.$$

Applying a result in reference [6], namely that the solution of this equation can be written as

$$p(t;\, t_f)=X^T(t_f,\, t)\psi_x^T(x^*(t_f))\, v \tag{2.13}$$

where $X(t,s)$ is defined by

$$\dot{X}(t,\, s)=A(t)X(t,\, s)+B(t)X(t-\tau,\, s), \qquad \text{for } (t,s)\in [s,t_f]\times[t_0,t_f]$$

$$X(s,s)=I, \quad X(t,s)=0 \ \text{ for } (t,s)\in[t_0-\tau,s]\times[t_0,t_f].$$

in addition, we require that $p(t,\, t_f)$ satisfies (2.12). We shall consider the matrix

$$V(t_1,\, t_f)=\psi_x(x^*(t_f))\int_{t_1}^{t_f} X(t_f,\, t)f_u(t)f_u^T(t)X^T(t_f,\, t)dt\,\psi_x^T(x^*(t_f)).$$

Theorem 2.2: A necessary and sufficient condition for an extremal arc satisfying $p(t_f)=\psi_x^T(x^*(t_f))v$ to be normal on $[t_1,\, t_f]$ is that the $r\times r$ matrix $V(t_1,\, t_f)$ be non-singular.

Proof: suppose $\mathcal{C}$ is abnormal on $[t_1,\, t_f]$. Then there exists non-vanishing vectors $p(t)$, v satisfying (2.9) - (2.12) can be represented by (2.13),

$$p(t;\, t_f)=X^T(t_f,\, t)\psi_x^T(x^*(t_f))v$$

and $p(t)$ is non-trivial if and only if v is non-trivial. Thus, if $\mathcal{C}$ is abnormal on $[t_1, t_f]$, equation (2.12) yields

$$f_u^T(t)X^T(t_f, t)\psi_x^T(x^*(t_f))v=0$$

on $[t_1, t_f]$ for some non-vanishing vector v, except finite points. Hence

$$v^T V(t_1, t_f)v=0. \tag{2.14}$$

Since $V(t_1, t_f)$ is non-negative definite, (2.14) implies that V is singular.

Conversely, suppose now that the matrix $V(t_1, t_f)$ is singular. Then there exists a non-vanishing vector v satisfying (2.14) or

$$\int_{t_1}^{t_f} |f_u^T(t)X^T(t_f, t)\psi_x^T(x^*(t_f))v|^2\, dt=0$$

this implies that

$$f_u^T(t)X^T(t_f, t)\psi_x^T(x^*(t_f))v=0, \tag{2.15}$$

at all but a finite number of points on $[t_1, t_f]$. Since solution of the equations (2.9), (2.10) and (2.11) are represented by

$$p(t, t_f)=X^T(t_f, t)\psi_x^T(x^*(t_f))v$$

(2.15) implies that (2.12) is satisfied. Since $\psi_x(x^*(t_f))$ has full rank, p(t) does not vanish, which implies $\mathcal{C}$ is abnormal on $[t_1, t_f]$. Thus, $\mathcal{C}$ is normal on $[t_1, t_f]$ if and only if $V(t_1, t_f)$ is non-singular.

3. The Relationship between Normality and Controllability

Combining the theorem of the preceding two sections, we have the following theorem.

Theorem 2.3: A necessary and sufficient condition for an extremal arc $\mathscr{C}=(x^*, u^*)$ satisfing $p(t_f)=\psi_x^T(x^*(t_f))v$ to be normal on $[t_0, t_f]$ is that the linearized variational equation

$$\dot{\xi}=f_x(t)\xi+f_{x_\tau}(t)\xi(t-\tau)+f_u(t)\eta, \quad \text{for } t\in[t_0, t_f]$$

$$\xi_0(t)=0 \qquad \text{for } t\in[t_0-\tau, t_0]$$

is controllable to the hyperplane $\psi_x(x^*(t_f))\xi=0$ on this time interval.

Theorem 2.4: The order of abnormality of $\mathscr{C}$ on $[t_1, t_f]$ is r-q where q is the rank of the matrix $V(t_1, t_f)$.

Proof: At first, we suppose that q=r, which implies $V(t_1, t_f)$ is non-singular, by Theorem 2.2, the order of abnormality of $\mathscr{C}$ is zero. Hence is normal.

Next, we suppose the order of abnormality of $\mathscr{C}$ on $[t_1, t_f]$ is k, where $1\le k<r$. Let $v^1, \dots, v^r$ be linearly independent in E^r. Let Z(t) be an nxr matrix whose columns z^i are linearly independent solutions of

$$\dot{z}=-f_x^T(t)z-f_{x_\tau}^T(t+\tau)z(t+\tau), \qquad t_0\le t\le t_f-\tau \tag{2.16}$$

$$\dot{z}=-f_x^T(t)z, \qquad t_f-\tau\le t\le t_f \tag{2.17}$$

satisfying $z^i(t_f)=\psi_x^T(x^*(t_f))v^i \qquad i=1, \dots\dots, r\ .$ (2.18)

We may suppose that the last k columns of Z satisfy

$$f_u^T(t)z^i=0 \tag{2.19}$$

at all but a finite number of points of $[t_i, t_f]$.

Let $\quad w^i = f_u^T z^i, \qquad i=1, \dots, r$

then, $\qquad w^i=0, \quad$ for $i=r-k+1, \dots, r$.

We say that the functions $w^1, \dots, w^{r-k}$ must be linearly independent on $[t_i, t_f]$. For if they are not then there exist constants $c_1, \dots, c_r$, not all zero, such that $\qquad \sum_{i=1}^{r-k} c_i w^i = 0, \qquad t \in [t_i, t_f]$.

Then, $\quad \bar{z} = \sum_{i=1}^{r-k} c_i z^i \quad$ would be a solution of (2.16)-(2.19) and $\bar{z}, z^{r-k+1}, \dots, z^r$ would be a set of k+1 linearly independent solution of (2.16)-(2.19) in contradition of the hypothesis. Thus the rxr matrix whose (i, j)th entry is

$\int_{t_i}^{t_f} w^i(t)^T w^j(t)dt, \qquad i, j=1, \dots, r \quad$ (since $w^1, \dots, w^{r-k}$ linearly independent)

has rank r-k, and this matrix can be written as follows

$$\left[\int_{t_i}^{t_f} w^i(t)^T w^j(t)dt\right] = \left[\int_{t_i}^{t_f} z^i(t)^T f_u(t) f_u^T(t) z^j(t)dt\right]$$

$$= \left[v^{i^T} \psi_x(x^*(t_f)) \int_{t_i}^{t_f} X(t_f, t) f_u(t) f_u^T(t) X^T(t_f, t)dt\, \psi_x^T(x^*(t_f)) v^j\right] = M^T V(t_i, t_f) M$$

where M is an rxr non-singular matrix. Hence $V(t_i, t_f)$ has rank $q=r-k$. This proves the theorem.

CHAPTER III

LOCAL DUAL PROPERTY

1. Introducton

The this chapter, we study an interesting local dual property concerning the functional,

$$J(u)=\int_{t_0}^{t_f} L(t, x, x_\tau, u)dt+\phi(x(t_f)) \tag{3.1}$$

subject to

$$\dot{x}=f(t, x, x_\tau, u), \qquad t_0 \le t \le t_f \tag{3.2}$$

$$x=x_0(t), \qquad t_0-\tau \le t \le t_0 \tag{3.3}$$

where x_0 is a given function and $x(t_f)$ is arbitrary (c.f.[7]).

Define the Hamiltonian $H=L+p^T f$

then $J(u)=\int_{t_0}^{t_f} [H-p^T\dot{x}]dt+\phi(x(t_f))$.

The first variation can be written as follows:

$$\delta J(u)=\int_{t_0}^{t_f} [H_x\delta x+H_{x_\tau}\delta x_\tau+H_u\delta u-p^T\delta\dot{x}]dt+\phi_{x_f}\delta x_f \tag{3.4}$$

let $\delta J(u)=0$, we obtain the Euler-Lagrange equations (1.5), (1.6), (1.7) and (1.8) (refer to p9).

Let us suppose that we have determined functions x(t), u(t) and p(t) that meet all the first-order necessary conditions. Now we look at neighboring extremal paths of x(t), u(t) and p(t), and investigate the second variation of J.

2. Second Variation

$$\delta^2 J(u, p)=\frac{1}{2}\int_{t_0}^{t_f}(2W+H_x\delta^2 x+H_{x_\tau}\delta^2 x_\tau+H_u\delta^2 u+H_p\delta^2 p-x\,\delta^2 p-2\delta\dot{x}^T\delta p-p^T\delta^2\dot{x})dt$$

$$+\frac{1}{2}\delta x_f^T\phi_{x_f x_f}\delta x_f+\frac{1}{2}\phi_{x_f}\delta^2 x_f$$

$$=\frac{1}{2}\int_{t_0}^{t_f}(2W-2\delta\dot{x}^T\delta p)dt+\frac{1}{2}\int_{t_0}^{t_f}(H_p-\dot{x})\,\delta^2 p\,dt+\frac{1}{2}\int_{t_0}^{t_f}(H_x\,\delta^2 x+H_{x_\tau}\delta^2 x_\tau+H_u\,\delta^2 u-p^T\delta^2\dot{x})dt$$

$$+\frac{1}{2}\delta x_f^T\phi_{x_f x_f}\delta x_f+\frac{1}{2}\phi_{x_f}\delta^2 x_f \tag{3.5}$$

where

$$2W=(\delta^T x,\ \delta^T x_\tau,\ \delta^T u,\ \delta^T p)\begin{pmatrix} H_{xx} & H_{xx_\tau} & H_{xu} & f_x \\ H_{x_\tau x} & H_{x_\tau x_\tau} & H_{x_\tau u} & f_{x_\tau} \\ H_{ux} & H_{ux_\tau} & H_{uu} & f_u \\ f_x^T & f_{x_\tau}^T & f_u^T & 0 \end{pmatrix}\begin{pmatrix}\delta x \\ \delta x \\ \delta u \\ \delta p\end{pmatrix} \tag{3.6}$$

Since $\dot{x}=f(t, x, x_\tau, u)=H_p$, then the second term in (3.5) vanishes. The third term can be written as

$$\frac{1}{2}\int_{t_0}^{t_f}(H_x\,\delta^2 x+H_{x_\tau}\delta^2 x_\tau+H_u\,\delta^2 u-p^T\delta^2\dot{x})dt$$

$$=\frac{1}{2}\int_{t_0}^{t_f-\tau}([H_x+H_{x_\tau}(t+\tau)+\dot{p}^T]\,\delta^2 x+H_u\,\delta^2 u)dt+\frac{1}{2}\int_{t_f-\tau}^{t_f}((H_x+\dot{p}^T)\,\delta x+H_u\delta^2 u)dt-\frac{1}{2}p^T(t_f)\delta^2 x_f \tag{3.7}$$

using (1.5), (1.6), (1.7) and (1.8), we have,

$$\delta^2 J(u, p)=\frac{1}{2}\int_{t_0}^{t_f}(2W-2\delta^T\dot{x}\,\delta p)+\frac{1}{2}\delta x_f^T\phi_{x_f x_f}\delta x_f \tag{3.8}$$

3. A Local Dual Property

From (3.2), we have,

$$\delta\dot{x}=f_x\,\delta x+f_{x_\tau}\delta x+f_u\,\delta u \tag{3.9}$$

for an neighboring extremal with δx, δx_τ, δu satisfying only (3.9), so that (3.8) may be simplified to

$$\delta^2 J_1=\frac{1}{\alpha}\int_{t_0}^{t_f}(\delta^T x,\ \delta^T x_\tau,\ \delta^T u,\ \delta^T p)\begin{pmatrix} H_{xx} & H_{xx_\tau} & H_{xu} & 0 \\ H_{x_\tau x} & H_{x_\tau x_\tau} & H_{x_\tau u} & 0 \\ H_{ux} & H_{ux_\tau} & H_{uu} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix}\delta x \\ \delta x_\tau \\ \delta u \\ \delta p\end{pmatrix}dt+\frac{1}{\alpha}\delta x_f^T\Phi_{x_f x_f}\delta x_f \tag{3.10}$$

Next, we can re-write (3.8) as follows:

$$\delta^2 J_2=\frac{1}{\alpha}\int_{t_0}^{t_f}[2W+2\,\delta^T x\ \delta\dot{p}]+\frac{1}{\alpha}\delta x_f^T\Phi_{x_f x_f}\delta x_f-\delta x_f^T\,\delta p(t_f) \tag{3.11}$$

On the other hand, for an neighboring extremal with δx, δx_τ, δu, δp, satisfying only (1.16), (1.17) and (1.15), we have,

$$\begin{aligned}\int_{t_0}^{t_f}2\,\delta x^T\delta\dot{p}dt&=-2[\int_{t_0}^{t_f-\tau}\delta x^T\delta(H_x+H_{x_\tau}(t+\tau))dt+\int_{t_f-\tau}^{t_f}\delta x^T\delta H_x dt]\\
&=-2[\int_{t_0}^{t_f}\delta x^T\delta H_x dt+\int_{t_0+\tau}^{t_f}\delta x^T(t-\tau)\,\delta H_{x_\tau}(t)dt]\\
&=-2[\int_{t_0}^{t_f}(\delta x^T\delta H_x+\delta x_\tau^T\delta H_{x_\tau})dt]\\
&=-2\int_{t_0}^{t_f}[\delta x^T H_{xx}\delta x+2\delta x^T H_{xx_\tau}\delta x_\tau+\delta x^T H_{xu}\delta u+\delta x_\tau^T H_{x_\tau u}\delta u+\delta x_\tau^T H_{x_\tau x_\tau}\delta x_\tau+\delta x^T f_x^T\delta p\\
&\quad+\delta x_\tau^T f_{x_\tau}^T\delta p]dt\end{aligned} \tag{3.12}$$

substituting (3.12) into (3.11), we obtain

$$\delta^2 J_2 = \frac{1}{\alpha}\int_{t_0}^{t_f} [2W - 2\delta^T x H_{xx}\delta x - 4\delta^T x H_{xx_\tau}\delta x_\tau - 2\delta^T x H_{xu}\delta u - 2\delta^T x_\tau H_{x_\tau u}\delta u - 2\delta^T x\, f_x^T \delta p - 2\delta^T x_\tau f_{x_\tau}^T \delta p]dt$$

$$-\frac{1}{\alpha}\delta x^T(t_f)\,\phi_{x_f x_f}\delta x(t_f)$$

$$= \frac{1}{\alpha}\int_{t_0}^{t_f} [(\delta^T x, \delta^T x_\tau, \delta^T u, \delta^T p) \begin{pmatrix} -H_{xx} & -H_{xx_\tau} & -H_{xu} & 0 \\ -H_{x_\tau x} & -H_{x_\tau x_\tau} & -H_{x_\tau u} & 0 \\ -H_{ux} & -H_{ux_\tau} & -H_{uu} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta x_\tau \\ \delta u \\ \delta p \end{pmatrix} dt$$

$$+2(\delta x^T H_{xu} + \delta x_\tau^T H_{x_\tau u} + \delta u^T H_{uu} + \delta p^T f_u)\delta u]dt - \frac{1}{\alpha}\delta x^T(t_f)\,\phi_{x_f x_f}\delta x(t_f)$$

using (1.15), we have

$$\delta^2 J_2 = \frac{1}{\alpha}\int_{t_0}^{t_f} (\delta^T x, \delta^T x_\tau, \delta^T u, \delta^T p) \begin{pmatrix} -H_{xx} & -H_{xx_\tau} & -H_{xu} & 0 \\ -H_{x_\tau x} & -H_{x_\tau x_\tau} & -H_{x_\tau u} & 0 \\ -H_{ux} & -H_{ux_\tau} & -H_{uu} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta x_\tau \\ \delta u \\ \delta p \end{pmatrix} dt - \frac{1}{\alpha}\delta x^T(t_f)\,\phi_{x_f x_f}\delta x(t_f) \qquad (3.13).$$

Comparing (3.13) with (3.10), we observe that the elements in (3.13) have exactly the opposite signs as compared to those of (3.10). Thus, we obtain an interesting property of the second variation of J(u).

APPENDIX

EFFECT OF ALLOWING DELAY TO BE FUNCTION OF TIME

Let us consider $J(u)=\int_{t_0}^{t_f}(H-p^T\dot{x})dt+\phi(x_f)$, no corners.

it is convenient to introduce the notation $\tau=\tau(t)$, $\tau_0=\tau(t_0)$, $\tau_f=\tau(t_f)$, and $\dot{x}=\frac{d}{dv}x(v)$, where $v=t-\tau(t)$. It is assumed that $\dot{\tau}<1$; this assumption implies that $t-\tau(t)$ is monotonically increasing.

$$\delta J=\phi_{x_f}\delta x_f+\int_{t_0}^{t_f}(H_x\xi+H_{x_\tau}\xi_\tau+H_u\eta+\dot{p}^T\xi)dt-p^T\xi\Big|_{t_0}^{t_f}$$

$$=(\phi_{x_f}-p(t_f))\delta x_f+\int_{t_0}^{t_f}[(H_x+\dot{p})\xi+H_{x_\tau}\xi_\tau+H_u\eta]\,dt$$

we consider $\int_{t_0}^{t_f}H_{x_\tau}\xi_\tau dt$: let $t-\tau(t)=s$, then $(1-\dot{\tau}(t))\,dt=ds$

$$\int_{t_0-\tau_0}^{t_f-\tau_f}\frac{H_{x_\tau}}{1-\dot{\tau}}\Big|^{t}\xi\,ds=\int_{t_0}^{t_f-\tau_f}\frac{H_{x_\tau}}{1-\dot{\tau}}\Big|^{s}\xi\,dt \qquad (*)$$

where little s is the solution of $s-\tau(s)=t$.

Substituting $(*)$ into δJ, we have

$$\delta J=(\phi_{x_f}-p(t_f))\delta x_f+\int_{t_0}^{t_f-\tau_f}(H_x+\dot{p}^T+\frac{H_{x_\tau}}{1-\dot{\tau}}\Big|^{s})\xi dt+\int_{t_f-\tau_f}^{t_f}(H_x+\dot{p}^T)\xi dt+\int_{t_0}^{t_f}H_u\eta\,dt$$

if $\delta J=0$, we obtain the following equations

$$\phi_{x_f}=p(t_f)$$

$$\dot{p}^T=-H_x-\frac{H_{x_\tau}}{1-\dot{\tau}}\Big|^{s} \qquad t_0\le t\le t_f-\tau_f$$

$$\dot{p}^T=-H_x \qquad t_f-\tau_f\le t\le t_f$$

$$H_u=0 \qquad t_0\le t\le t_f$$

where s is the solution of $s-\tau(s)=t$.

Next, we calculate $\delta^2 J$:

$$\delta^2 J=\frac{1}{2}\delta x_f^T \phi_{x_f x_f}\delta x_f+\frac{1}{2}\phi_{x_f}\delta^2 x_f+\frac{1}{2}\int_{t_0}^{t_f}2W dt-\frac{1}{2}p^T\delta^2 x\Big|_{t_0}^{t_f}+\frac{1}{2}\int_{t_0}^{t_f-\tau_f}\left(H_x+\frac{H_{x_\tau}}{1-\dot{\tau}}\Big|^{s}+\dot{p}^T\right)\delta^2 x dt$$

$$+\frac{1}{2}\int_{t_f-\tau_f}^{t_f}(H_x+\dot{p})\delta^2 x dt+\frac{1}{2}\int_{t_0}^{t_f}H_u\delta^2 u dt.$$

Since it satisfies the first-order necessary conditions, the coefficients of $\delta^2 u, \delta^2 x, \delta^2 x_\tau$ vanish and $\delta^2 J=\frac{1}{2}\delta x_f^T \phi_{x_f x_f}\delta x_f+\frac{1}{2}\int_{t_0}^{t_f}2W dt$, where $2W$ is defined as before.

So many results which we obtained before are also true when τ is a function of time t.

REFERENCES

1. William John, Optimal Control Problems and Differential Games with Time Delays. Ph. D Thesis, Department of Engineering, mechanical, Northwestern University. 1971.

2. David K. Hughes, Variatinal and Optimal Control Problem with Delay Argument, Journal of Optimization Theory and Applications: Vol.2, No.1, pp 1-15. 1968.

3. Violet B. Haas, On the Singular Bolza Problem. Advances in Mathematics, 24, pp189-205. 1977.

4. Violet B. Haas, The Clebsch and Jacobi Conditions for Singular Extremals, Int. J. Control. Vol.27, No.4, pp557-570. 1978.

5. Jacobson, D. H. and Bell, D. J. Singular Optimal Control Problems, Academic Press. New York. 1975.

6. M. C. Delfour and S. K. Mitter, Controllability, Observability and Optimal Feedback Control of Affine Hereditary Differential System, SIAM, J. Control, Vol.10, No.2, pp298-328. 1972.

7. W. L. Chan and C. H. Tai, Dual Variational Priniciples with Deviating Argement and Discontinuous Extremals, Journal of Optimization Theory

and Applications: Vol.23, No.4, pp593-604. 1977.

8. Violet B. Haas, Normality and Controllability for the Problem of Bolza, Int. J. Control. Vol.30, No.2, pp331-338. 1979.

9. M. A. Connor, Singular Control of Delay Systems, Journal of Optimization Theory and Application: Vol.13, No.5, pp538-543. 1974.

10. Violet B. Hass, On Normality and Conjugate Point Criteria for Singular Extremals, SIAM. J. Control, Vol.13, No.6, pp1172-1182. 1975.

11. A Ben-Isreal, T. N. E. Greville, Generalized Inverses, Theory and Applications. Wiley, New York. 1974.

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